ON THE ADDITION OF CONVEX CURVES

ON THE ADDITION OF CONVEX CURVES. II

ON THE VALUES OF THE RIEMANN ζ-FUNCTION ON FIXED LINES $\sigma > 1$

A Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

RICHARD KERSHNER

JUNE, 1937

Reprinted from THE AMERICAN JOURNAL OF MATHEMATICS,
Vol. LVIII, No. 4, October, 1936, pp. 737-746,
Vol. LIX, No. 2, April, 1937, pp. 423-426,
Vol. LIX, No. 1, January, 1937, pp. 167-174.

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ACKNOWLEDGMENT

I am grateful for this opportunity to express to Professor Wintner my appreciation for his direction and advice.

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ON THE ADDITION OF CONVEX CURVES.

By Richard Kershner.

If C denotes a convex Jordan curve in the plane let $\Omega(C)$ denote the open, bounded domain surrounded by C, so that the closed region $\bar{\Omega}(C)$ is the logical sum $\Omega(C) + C$. If A and B are two point sets in the z-plane, where $z = x + iy$, their vectorial sum [1] will be denoted by $A\ (+)\ B$.

In connection with his investigation of the distribution of the values of the Riemann zeta function,[2] Bohr [3] has studied the vectorial addition of convex curves from a geometrical point of view. He has proven,[4] among other things, that if $C_1, C_2, \cdots, C_n$ are convex curves, then either there exists a convex C_E such that

$$C_1(+)\ C_2(+)\cdots(+)\ C_n = \bar{\Omega}(C_E) \tag{1}$$

or there exist two convex curves C_E and C_I such that $\Omega(C_E) \supset \Omega(C_I)$ and

$$C_1(+)\ C_2(+)\cdots(+)\ C_n = \bar{\Omega}(C_E) - \Omega(C_I) \tag{2}$$

so that the vectorial sum is either a closed bounded region or a ring shaped region bounded by two convex curves.

Following a suggestion of Wintner, Haviland [5] applied the supporting function (Stützfunktion) of Brunn and Minkowski to the study of Bohr's problem and showed that if $h_1(\theta), h_2(\theta), \cdots, h_n(\theta)$ are the supporting functions of the curves $C_1, C_2, \cdots, C_n$ and if $h_E(\theta)$ is the supporting function of C_E, then

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[1] By the vectorial sum of two sets A and B is meant the set $z = z_a + z_b$, where $z_a \subset A$ and $z_b \subset B$. It is clear that this addition is associative and commutative.

[2] Cf. e. g. Titchmarsh, *The Zeta Function of Riemann.*

[3] H. Bohr, "Om Addition of unendelig mange konvekse Kurver," *Danske Videnskabernes Selskab* (Forhandlinger, 1913), pp. 325-366; also, for a further study cf. H. Bohr and B. Jessen, "Om Sandsynlighedsfordelinger ved Addition of konvekse Kurver," *Danske Videnskabernes Selskab, Skrifter,* (8), vol. 12, no. 3 (1929).

[4] Cf. H. Bohr, *loc. cit.*; also H. Bohr and B. Jessen, *loc. cit.* For a short presentation of the proof of this fact cf. B. Jessen and A. Wintner, "Distribution functions and the Riemann zeta function," *Transactions of the American Mathematical Society,* vol. 38 (1935), p. 69.

[5] E. K. Haviland, "On the addition of convex curves in Bohr's theory of Dirichlet series," *American Journal of Mathematics,* vol. 55 (1933), pp. 332-334.

$$h_E(\theta) = \sum_{j=1}^{n} h_j(\theta). \tag{3}$$

Using this explicit formula and Minkowski's results on mixed area, Haviland improved on the results of Bohr concerning relations between the areas of $\Omega(C_j)$ and $\Omega(C_E)$. He also derived results with regard to lengths and curvatures. Finally, Bohr and Jessen [6] have recently shown that, with the use of the supporting functions, it is possible to determine the set of those $\sigma > 1$ for which the closure of the values attained by the logarithm of the Riemann zeta function on the line σ is a convex region; for the remaining values of $\sigma > 1$ this closure is ring-shaped. They also pointed out a geometrical criterion for the existence of an inner curve in the case of arbitrary symmetric convex curves.

The object of the present note is an analytical discussion of the inner curve C_I similar to that given by Haviland for C_E. In particular there will be obtained a rule giving the supporting function $h_I(\theta)$ of C_I in terms of the supporting functions $h_j(\theta)$ of the added curves C_j in the same manner as the rule (2) gives $h_E(\theta)$. There will also be obtained results for the length of C_I, for the curvature along this curve and for the area of $\Omega(C_I)$. As an application of the analytical results concerning $h_I(\theta)$ there is given a simple mechanical construction of C_I whenever C_I exists.[7] In particular there is a geometrical criterion that C_I exist, i. e. that (2) hold. This criterion will imply the above-mentioned criterion found by Bohr and Jessen in the case of symmetric curves.

We treat first the sum of two convex curves and prove

THEOREM I_0. *Let $n = 2$ so that either $C_1 (+) C_2 = \bar{\Omega}(C_E)$ or*

$$C_1 (+) C_2 = \bar{\Omega}(C_E) - \Omega(C_I).$$

It may be supposed that $l_1 \geqq l_2$ where l_j is the length of C_j. Let $\tilde{C}_2$ represent the curve obtained by rotating C_2 about the origin of the plane through an angle π. Then the curve C_I exists if and only if $\tilde{C}_2$ may be placed in $\Omega(C_1)$ by a translation.

Proof. Suppose that C_I exists so that $\Omega(C_I)$ is not empty. It is clear that a translation of C_2 effects only a translation on the region $\bar{\Omega}(C_E) - \Omega(C_I)$.

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[6] H. Bohr and B. Jessen, "On the distribution of the values of the Riemann zeta function," *American Journal of Mathematics*, vol. 58 (1936), pp. 35-44.

[7] For further applications cf. R. Kershner and A. Wintner, "On the boundary of the range of values of $\zeta(s)$," *American Journal of Mathematics*, vol. 58 (1936), pp. 421-425.

Hence we may translate C_2 into a new position C_2^* in such a way that $\Omega(C_I^*)$ includes the origin of the plane, where, of course,

$$C_1(+)C_2^* = \bar{\Omega}(C_E^*) - \Omega(C_I^*).$$

Since the origin is in $\Omega(C_I^*)$, no point of C_2^* is symmetrical, with respect to the origin, to a point of C_1. Hence, the curve $\tilde{C}_2^*$, obtained by rotating C_2^* about the origin through an angle π, does not intersect C_1. Since $l_2 \leqq l_1$, either $\tilde{C}_2^*$ is in $\Omega(C_1)$ or $\bar{\Omega}(C_1)$ and $\bar{\Omega}(\tilde{C}_2^*)$ are disjoint. Now the latter case is impossible. For in this case the set $\bar{\Omega}(C_1)\ (+)\ \bar{\Omega}(C_2^*)$ would not contain the origin, whereas it is obvious [8] that

$$\bar{\Omega}(C_1)\ (+)\ \bar{\Omega}(C_2^*) = \bar{\Omega}(C_E^*)$$

and we have supposed that the origin was in $\Omega(C_I^*) \subset \Omega(C_E^*)$. Hence $\tilde{C}_2^*$ is in $\Omega(C_1)$ and the "only if" of Theorem I_0 is proven. The "if" may be shown by following the above proof in the reverse direction.

Next we prove

THEOREM, II_0. *Let $n = 2$ and $l_1 \geqq l_2$. Let T_I be the set of all points (x, y) of the plane which satisfy*

$$x\cos\theta + y\sin\theta < h_1(\theta) - h_2(\theta + \pi) \quad \textit{for all } \theta. \tag{4}$$

Then if T_I is not empty C_I exists and $T_I = \Omega(C_I)$; if T_I is empty C_I does not exist.

Proof. It is clear, here also, that a translation of C_2 effects a translation of T_I. Hence we may translate C_2 into a new position C_2^* in such a way that T_I^* contains the origin (where, of course, T_I^* is defined in terms of C_1 and C_2^* exactly as T_I was defined in terms of C_1 and C_2) if and only if T_I is not empty. But T_I^* contains the origin if and only if

$$0 < h_1(\theta) - h_2^*(\theta + \pi) \text{ for all } \theta \tag{5}$$

where $h_2^*(\theta)$ is the supporting function of C_2^*. Since the supporting function $h(\theta)$ of any convex curve is defined [9] as

$$h(\theta) = \max\,[x\cos\theta + y\sin\theta], \qquad (x, y) \text{ on } C, \tag{6}$$

it is clear that $h_2^*(\theta + \pi)$ is the supporting function of the curve $\tilde{C}_2^*$

[8] Cf. E. K. Haviland, *loc. cit.*, p. 332.

[9] Cf., e. g., G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis I.* (Berlin, 1925), p. 106.

obtained by rotating C_2^* about the origin through an angle π. Hence the relation (5) implies that $\bar{C}_2^*$ is within C_1. Thus T_I is not empty if and only if $\bar{C}_2$ may be placed entirely within C_1 by means of a translation. Consequently, by Theorem I_0, T_I is empty if and only if C_I does not exist and the last part of Theorem II has been proven.

Suppose now that T_I is not empty.

First, $T_I \subset \bar{\Omega}(C_E)$. For, it is seen from (3) and (6) that $\bar{\Omega}(C_E)$ is the set of points (x, y) satisfying

$$x \cos\theta + y \sin\theta \leqq h_1(\theta) + h_2(\theta) \text{ for all } \theta; \tag{7}$$

and (4) implies (7), since $-h_2(\theta+\pi) < h_2(\theta)$. In fact, $h_2(\theta) + h_2(\theta+\pi)$ (> 0) is precisely the width of the curve C_2 in the direction θ.

Next $T_I \subset \Omega(C_I)$. (It has already been shown that C_I exists). For let $z_j = x_j + iy_j$ be a point of C_j, where $j = 1, 2$, and let $\theta_1 = \theta(z_1)$ be a value of θ such that

$$x_1 \cos\theta_1 + y_1 \sin\theta_1 = h_1(\theta_1). \tag{8}$$

Then, since z_2 is on C_2, the definition (6) of $h_2(\theta)$ gives

$$x_2 \cos\theta + y_2 \sin\theta \leqq h_2(\theta) \text{ for all } \theta; \tag{9}$$

hence, in particular (9) holds for $\theta = \theta_1 + \pi$ and

$$x_2 \cos\theta_1 + y_2 \sin\theta_1 \geqq -h_2(\theta_1 + \pi). \tag{10}$$

From (8) and (10)

$$(x_1 + x_2)\cos\theta_1 + (y_1 + y_2)\sin\theta_1 \geqq h_1(\theta_1) - h_2(\theta_1 + \pi),$$

so that $C_1 (+) C_2 = \bar{\Omega}(C_E) - \Omega(C_I) \subset \bar{\Omega}(C_E) - T_I$, i. e. $T_I \subset \Omega(C_I)$.

Finally, $T_I \supset \Omega(C_I)$. For, let B be the boundary of T_I and let $z_0 = x_0 + iy_0$ be any point of B. Then, by the definition of T_I,

$$x_0 \cos\theta + y_0 \sin\theta \leqq h_1(\theta) - h_2(\theta + \pi) \text{ for all } \theta, \tag{11}$$

while, for at least one θ, say $\theta = \theta_0$,

$$x_0 \cos\theta_0 + y_0 \sin\theta_0 = h_1(\theta_0) - h_2(\theta_0 + \pi). \tag{12}$$

Let $z_2 = z_2(\theta_0) = x_2(\theta_0) + iy_2(\theta_0)$ be a point of C_2 in the direction $\theta_0 + \pi$, i. e. satisfying

$$x_2 \cos(\theta_0 + \pi) + y_2 \sin(\theta_0 + \pi) = h_2(\theta_0 + \pi). \tag{13}$$

From (6),

$$x_2 \cos(\theta + \pi) + y_2 \sin(\theta + \pi) \leqq h_2(\theta + \pi) \text{ for all } \theta. \tag{14}$$

Let $z_0 - z_2 = z_1 = x_1 + iy_1$. Then, from (12), (13) and (11), (14)

$$x_1 \cos\theta_0 + y_1 \sin\theta_0 = h_1(\theta_0) \text{ and } x_1 \cos\theta + y_1 \sin\theta \leqq h_1(\theta) \text{ for all } \theta.$$

Thus z_1 is a point of C_1, as well as z_2 a point of C_2, and $z_1 + z_2 = z_0$ where z_0 was any point of B. Thus, $B \subset \bar{\Omega}(C_E) - \Omega(C_I)$. Since T_I is obviously convex, this implies $T_I \supset \Omega(C_I)$. But it has already been shown that $T_I \subset \Omega(C_I)$ so that the proof of Theorem II_0 is complete.

Theorem II_0 implies a simple mechanical way in which $\Omega(C_I)$ may be determined: $\Omega(C_I)$ is, up to a translation, the locus of a point which is considered as rigidly attached to C_2 when C_2 is translated in all possible manners in $\Omega(C_1)$. This fact, which is completely equivalent to Theorem II_0, merely expresses the amount of indeterminateness in the choice of the translation which occurred in the proof of Theorem I_0. It would be possible to give an entirely geometrical proof of Theorem II_0 based on this reasoning.

Since $B = C_I$, every point z_0 of C_I satisfies (11) and (12). From these inequalities it does not follow that the supporting function $h_I(\theta)$ of C_I is identically $h_1(\theta) - h_2(\theta + \pi)$. However (11) gives immediately

$$h_I(\theta) \leqq h_1(\theta) - h_2(\theta + \pi) \text{ for all } \theta.$$

Furthermore, if there is a unique supporting line through a given point z_0 of C_I, i. e. if

$$x_0 \cos\theta + y_0 \sin\theta = h_I(\theta)$$

is satisfied for a unique $\theta = \theta_0$, then by (12)

$$h_I(\theta_0) = h_1(\theta_0) - h_2(\theta_0 + \pi).$$

The curve C_I is said to have a corner at $z = z_0$ if the supporting line through z_0 is not unique. The above remarks imply

THEOREM III_0. *The supporting function $h_I(\theta)$ of C_I satisfies*

$$h_I(\theta) \leqq h_1(\theta) - h_2(\theta + \pi) \text{ for all } \theta \tag{15}$$

and the equality sign holds save at most for the θ-intervals corresponding to corners of C_I.

We now give the corresponding theorems for the sum of n convex curves.

THEOREM I. *Let $C_1, C_2, \cdots, C_n$ be n convex curves of lengths $l_1, l_2, \cdots, l_n$ respectively and call $C_1 (+) C_2 (+) \cdots (+) C_n = \bar{\Omega}(C_E) - \Omega(C_I)$ where $\Omega(C_I)$ represents the empty set if C_I does not exist. Suppose that $l_1 \geqq l_2 \cdots \geqq l_n$. Call $C_2 (+) C_3 (+) \cdots (+) C_n = \bar{\Omega}(D_E) - \Omega(D_I)$ where $\Omega(D_I)$ is the empty set if D_I does not exist. Then C_I exists if and only if $\bar{D}_E$, the curve obtained by rotating D_E about the origin through an angle π, can be placed in $\Omega(C_1)$ by a translation.*

THEOREM II. *Let T_I be the set of all points z satisfying*

$$x \cos\theta + y \sin\theta < h_1(\theta) - \sum_{j=2}^{n} h_j(\theta + \pi) \quad \text{for all } \theta, \tag{16}$$

where $h_j(\theta)$ is the supporting function of C_j $(j = 1, 2, \cdots, n)$. Then if T_I is not empty C_I exists and $T_I = \Omega(C_I)$; if T_I is empty C_I does not exist.

Proof. Since the supporting function $g_E(\theta)$ of D_E satisfies, by (3), the relation

$$g_E(\theta) = \sum_{j=2}^{n} h_j(\theta), \tag{17}$$

it may be shown, as it was in the case of two curves, that Theorem I is implied by Theorem II; so it is sufficient to prove the latter.

Now Theorem II is true for the case of two curves. Suppose Theorem II has been proven for the case of $n - 1$ curves. Then, using the notations of Theorem I and calling $g_I(\theta)$ the supporting function of D_I, we have

$$g_I(\theta) \leqq h_2(\theta) - \sum_{j=3}^{n} h_j(\theta + \pi). \tag{18}$$

Now it is well known that if $h(\theta)$ is the supporting function and l the length of a convex curve,

$$l = \int_0^{2\pi} h(\phi)\, d(\phi). \tag{19}$$

Let λ_I be the length of D_I. Then, by (18) and (19),

$$\lambda_I \leqq l_2 - \sum_{j=3}^{n} l_j. \tag{20}$$

Suppose first that T_I is empty. Then, as in Theorem II_0, the curve $\bar{D}_E$ cannot be placed within C_1 by a translation. On the other hand, the length λ_I of D_I is, by (20), less than l_2, and, since $l_1 \geqq l_2$, less than l_1. Hence, there exists a

convex curve E between D_I and D_E, i. e. in $\bar{\Omega}(D_E) - \Omega(D_I)$, of length less than or equal to l_1 such that $\tilde{E}$ cannot be placed within C_1 by a translation. The region $C_1 (+) E$ is, by Theorem I_0, a closed convex region, and a fortiori, C_I does not exist.

Suppose now that T_I is not empty. Then $C_1 (+) D_E = \bar{\Omega}(C_E) - T_I$ by (17) and Theorem II_0. Let F be any convex curve in $\bar{\Omega}(D_E) - \Omega(D_I)$ and let $k(\theta)$ be its supporting function. Let $C_1 (+) F = \bar{\Omega}(G_E) - \Omega(G_I)$ where $\Omega(G_I)$ represents the empty set if G_I does not exist. Then the supporting function of G_E is, by (3), equal to $h_1(\theta) + k(\theta)$ and the supporting function of G_I is, by Theorem II_0, equal to $h_1(\theta) - k(\theta + \pi)$, except, at most, at the possible corners of G_I.

Since $k(\theta) \leqq g_E(\theta) = \sum_{j=2}^{n} h_j(\theta)$ for all θ, we have

$$\begin{aligned} h_1(\theta) - g_E(\theta + \pi) &\leqq h_1(\theta) - k(\theta + \pi) \\ &\leqq h_1(\theta) + k(\theta) \leqq h_1(\theta) + g_E(\theta) \quad \text{for all } \theta. \end{aligned}$$

Hence $\bar{\Omega}(G_E) - \Omega(G_I) \subset \Omega(C_E) - T_I$. But F was any convex curve in $\bar{\Omega}(D_E) - \Omega(D_I)$. Consequently

$$C_1 (+) D_E = C_1 (+) C_2 (+) \cdots (+) C_n \tag{21}$$

and $T_I = \Omega(C_I)$.

Relations (17) and (21), together with Theorem III_0, give immediately

THEOREM III. *The supporting function $h_I(\theta)$ of C_I satisfies*

$$h_I(\theta) \leqq h_1(\theta) - \sum_{j=2}^{n} h_j(\theta + \pi), \text{ for all } \theta, \tag{22}$$

and the equality sign holds save at most for the θ-intervals corresponding to corners of C_I. Thus, if $\{z_k\}$ is the sequence of corners (if any) of C_I and $\{\Theta_k\}$ is the sequence of the corresponding open θ-intervals and if Θ denotes the closed set $\Theta = [0, 2\pi] - \Sigma\Theta_k$ then

$$\begin{aligned} h_I(\theta) &= h_1(\theta) - \sum_{j=2}^{n} h_j(\theta + \pi), \text{ if } \theta \subset \Theta \\ h_I(\theta) &= x_j \cos\theta + y_j \sin\theta, \text{ if } \theta \subset \Theta_j;\ z_j = x_j + iy_j. \end{aligned}$$

It may be mentioned that all the above results are true, with appropriate changes in statement, for the vectorial sum of convex bodies of any number of dimensions.

We now proceed to the length, area and curvature relations implied by (22).

Formula (19), for the length of a convex curve, together with (22), gives immediately

$$l_I \leqq l_1 - \sum_{j=2}^{n} l_j \tag{23}$$

where l_I is the length of C_I.

It is well known that the area $A(C)$ of $\Omega(C)$, where C is a convex curve with supporting function $h(\theta)$, is represented by the formula

$$A(C) = \tfrac{1}{2}\int_0^{2\pi} [h^2(\phi) - (h'(\phi))^2]d\phi. \tag{24}$$

Let $M(C, D)$, where C and D are convex curves, represent the Minkowski mixed area [10] of the regions $\Omega(C)$ and $\Omega(D)$. Then by (22), (24), and (17)

$$A(C_I) \leqq \tfrac{1}{2}\int_0^{2\pi} [(h_1(\phi) - g_E(\phi+\pi))^2 - (h'_1(\phi) - g'_E(\phi+\pi))^2]d\phi$$

or

$$A(C_I) \leqq \tfrac{1}{2}\int_0^{2\pi} [h_1{}^2(\phi) - (h'_1(\phi))^2]d\phi + \tfrac{1}{2}\int_0^{2\pi} [g_E{}^2(\phi+\pi) - (g'_E(\phi+\pi))^2]d\phi$$
$$- \int_0^{2\pi} [h_1(\phi) g_E(\phi+\pi) - h'_1(\phi) g'_E(\phi+\pi)]d\phi$$

i. e.

$$A(C_I) \leqq A(C_1) + A(D_E) - 2M(C_1, D_E). \tag{25}$$

Since, according to Minkowski,

$$M(C, D) \geqq \sqrt{A(C)A(D)},$$

(25) becomes

$$\sqrt{A(C_I)} \leqq \sqrt{A(C_1)} - \sqrt{A(D_E)}$$

or, since [11] $\sqrt{A(D_E)} \geqq \sum_{j=2}^{n} \sqrt{A(C_j)}$,

$$\sqrt{A(C_I)} \leqq \sqrt{A(C_1)} - \sum_{j=2}^{n} \sqrt{A(C_j)}.$$

If (25) is compared with the corresponding formula of Haviland,

$$A(C_E) = A(C_1) + A(D_E) + 2M(C_1, D_E),$$

one finds

$$A = A(C_E) - A(C_I) \geqq 4M(C_1, D_E) = 4\sum_{j=2}^{n} M(C_1, C_j),$$

[10] Cf. e. g., W. Blaschke, *Kreis und Kugel* (Leipzig, 1916), pp. 106-107, or T. Bonnesen, *Les Problèmes des Isoperimètres* (Paris, 1929), Ch. V.

[11] Cf. E. K. Haviland, *loc. cit.*, p. 334.

where A is the area of the ring-shaped region (2). Hence

$$A \geqq 4\sqrt{A(C_1)} \sum_{j=2}^{n} \sqrt{A(C_j)}.$$

We summarize the above results on area and length relations in

THEOREM IV. *Let l_j be the length of C_j and $A(C_j)$ the area of $\Omega(C_j)$. Let l_I be the length of C_I and $A(C_I)$ the area of $\Omega(C_I)$. Let A be the area of $C_1 (+) C_2 (+) \cdots (+) C_n$. Then, if C_I exists,*

(i) $$l_I \leqq l_1 - \sum_{j=2}^{n} l_j$$

and

(ii) $$\sqrt{A(C_I)} \leqq \sqrt{A(C_1)} - \sum_{j=2}^{n} \sqrt{A(C_j)}.$$

Whether C_I exists or not

(iii) $$A \geqq 4 \sum_{j=2}^{n} M(C_1, C_j),$$

where $M(C_1, C_j)$ is the mixed area of $\Omega(C_1)$ and $\Omega(C_j)$, and

(iv) $$A \geqq 4\sqrt{A(C_1)} \sum_{j=2}^{n} \sqrt{A(C_j)}.$$

In (i) *and* (iii) *the equality sign holds if C_I has no corners.*

Suppose, now, that the supporting functions $h_j(\theta)$ have continuous second derivatives, and that the radius of curvature $\rho_j(\theta)$ of C_j is defined, so that

(26) $$\rho_j(\theta) = h_j(\theta) - h_j''(\theta).$$

Suppose that the point in the direction θ_1 on C_I is not a corner or a cluster point of corners, so that

$$h_I(\theta) = h_1(\theta) - \sum_{j=2}^{n} h_j(\theta + \pi), \text{ if } \theta_1 - \epsilon \leqq \theta \leqq \theta_1 + \epsilon$$

for some $\epsilon > 0$. Then, by (26),

$$\rho_I(\theta_1) = \rho_1(\theta_1) - \sum_{j=2}^{n} \rho_j(\theta_1 + \pi)$$

where $\rho_I(\theta_1)$ is the radius of curvature of C_I at the point in the direction θ_1. In particular, if C_I has no corners then $\rho_I(\theta) \geqq \sum_{j=2}^{n} \rho_j(\theta + \pi)$ for all θ where

the equality sign holds at most for discrete θ-values. It is easy to see, in view of Theorem II, that the converse of this last statement is also true. Thus we have

THEOREM V. *If $h_j(\theta)$ has a continuous second derivative for $j = 1, 2, \cdots, n$, then a necessary and sufficient condition that C_I exist and have no corners is that $\rho_1(\theta) \geqq \sum_{j=2}^{n} \rho_j(\theta + \pi)$ for all θ where the equality sign holds at most for discrete θ-values. In general, if the point in the direction θ_1 on C_I is not a corner or a cluster point of corners of C_I then the radius of curvature $\rho_I(\theta)$ of C_I is defined for $\theta = \theta_1$, and*

$$\rho_I(\theta_1) = \rho_1(\theta_1) - \sum_{j=2}^{n} \rho_j(\theta_1 + \pi).$$

THE JOHNS HOPKINS UNIVERSITY.

ON THE ADDITION OF CONVEX CURVES. II

By Richard Kershner.

By the vectorial sum $C_1(+)C_2$ of two convex curves C_1 and C_2 is meant the set of all points which may be represented in at least one way as the vectorial sum of a point on C_1 and a point on C_2. It has been shown by Bohr [1] that $C_1(+)C_2$ is either the closed interior of a convex curve C_E or is the closed annular region between two convex curves C_E and C_I, where C_I lies wholly within C_E. The outer boundary C_E of $C_1(+)C_2$ was discussed by Haviland,[2] who found very precise relationships between C_E and the component curves C_1, C_2 by the use of the Minkowski supporting function. For example, if C_1 and C_2 each possess a continuous positive radius of curvature then so does C_E and, in fact, if $\rho_1(\theta), \rho_2(\theta), \rho_E(\theta)$ are the radii of curvature of C_1, C_2, C_E respectively, at the point of C_1, C_2, C_E where the oriented normal has the inclination θ then $\rho_E(\theta) = \rho_1(\theta) + \rho_2(\theta)$.

Recently the author [3] has investigated the inner boundary curve C_I of $C_1(+)C_2$. The results obtained are similar to those of Haviland but the methods are essentially more complicated. The great distinction between the treatment and the results in the two cases is illustrated by the fact that the curve C_I may possess corners while both C_1 and C_2 are analytic.[4] This fact contrasts remarkably with the above remarks concerning the radius of curvature $\rho_E(\theta)$ of C_E.

The purpose of the present note is a discussion of the possible existence of corners in the curve C_I. Specifically, it will be shown that if C_1 and C_2 are analytic curves, then C_I can have but a finite number of corners. (It will be shown that this number may be arbitrarily large). On the other hand, if it be only required of C_1 and C_2 that they possess radii of curvature which

* Received February 28, 1937.

[1] H. Bohr, "Om Addition of uendelig mange konvekse Kurver," *Danske Videnskabernes Selskab* (Forhandlinger, 1913), pp. 325-366. For a short presentation of the proof of this fact cf. B. Jessen and A. Wintner, "Distribution functions and the Riemann zeta function," *Transactions of the American Mathematical Society*, vol. 38 (1935), p. 69.

[2] E. K. Haviland, "On the addition of convex curves in Bohr's theory of Dirichlet series," *American Journal of Mathematics*, vol. 55 (1933), pp. 332-334.

[3] R. Kershner, "On the addition of convex curves," *American Journal of Mathematics*, vol. 58 (1936), pp. 737-746.

[4] Cf., e. g., R. Kershner, "On the values of the Riemann ζ-function on fixed lines $\sigma > 1$," *American Journal of Mathematics*, vol. 59 (1937), pp. 167-174.

can be differentiated infinitely often, then it is possible that C_I have an infinite number of corners. Finally it will be shown that if C_1 and C_2 have each a continuous radius of curvature then the corners of C_I are nowhere dense.

In the sequel it will always be assumed that C_I exists. Then [5] one of the curves C_1, C_2 may be placed in the other, after a rotation through the angle π about the origin, by a translation. It will be assumed that C_1 is the "larger" of the two curves so that C_2 may be placed in C_1, in the manner indicated above. By a point of C_1, C_2, C_I, in the direction θ, or, briefly, a point θ, is meant a point where the oriented normal has the inclination θ. Every point of C_I, except a corner, has a direction in the cases to be considered. A corner of C_I will be said to have the direction (θ_1, θ_2) if θ_1 and θ_2 are respectively the lower and upper limits of the directions of points in the neighborhood of the corner.

Using these notations we prove

LEMMA I. *If C_I has a corner in the direction (θ_1, θ_2) and if C_1 and C_2 have continuous, positive radii of curvature $\rho_1(\theta)$ and $\rho_2(\theta)$ respectively, then $\rho_1(\theta) = \rho_2(\theta + \pi)$ for some θ in $\theta_1 < \theta < \theta_2$; and, on the other hand, if $\rho_1(\theta) \leqq \rho_2(\theta + \pi)$ for some interval $\theta'_1 < \theta < \theta'_2$ then C_I has a corner in the direction (θ_1, θ_2) where $\theta_1 \leqq \theta'_1 < \theta'_2 \leqq \theta_2$.*

Proof. In terms of the mechanical interpretation [6] of C_I it is clear that the existence of a corner of C_I in the direction (θ_1, θ_2) means that the curve C_2, after being rotated through an angle π about the origin, may be placed within C_1 in such a way that it has internal contact with C_1 at the two points of C_1 which have the directions θ_1, θ_2. Lemma I follows [7] immediately from this fact. Lemma I gives immediately

THEOREM I. *If C_1 and C_2 are analytic curves then C_I has at most a finite number of corners.*

For suppose C_I had an infinite number of corners. Then the function $\rho_1(\theta) - \rho_2(\theta + \pi)$, where $\rho_i(\theta)$ is the radius of curvature of C_i, would have zeros clustering as some particular point θ. But since $\rho_i(\theta)$ is regular analytic this would imply $\rho_1(\theta) \equiv \rho_2(\theta + \pi)$ and C_I would not exist.

Now if $\rho(\theta)$ is a positive, continuous, periodic function of θ, with period

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[5] R. Kershner, *loc. cit.* 3, Theorem I_0, p. 738.

[6] R. Kershner, *loc. cit.* 3, p. 741.

[7] Cf., e. g., S. Mukhopadhyaya, "Circles incident on an oval of undefined curvature," *Tôhoku Mathematical Journal*, vol. 34 (1931), pp. 115-129.

2π, then there will exist a closed convex curve of which $\rho(\theta)$ is the radius of curvature, if and only if the closure conditions

$$(1) \qquad \int_0^{2\pi} \rho(\theta) \cos\theta d\theta = 0; \qquad \int_0^{2\pi} \rho(\theta) \sin\theta d\theta = 0$$

are satisfied.[8] Using this fact it is very easy to show that the number of corners of C_I may be arbitrarily large even when both C_1 and C_2 are analytic. For let C_2 be a circle of radius r. Let

$$(2) \qquad \rho_1^{(n)}(\theta) = r + 1 + \cos n\theta - \delta_n, \qquad (n = 2, 3, 4, \cdots)$$

where $\delta_n > 0$ is chosen so small that, first of all, $\rho_1^{(n)}(\theta)$ is everywhere positive, so that ((1) being obviously satisfied) there does exist a closed convex curve $C_1^{(n)}$ having $\rho_1^{(n)}(\theta)$ as radius of curvature; and secondly that C_2 may be placed entirely within $C_1^{(n)}$. In order to satisfy the first requirement on δ_n it is obviously enough to choose $\delta_n < r$. To see that the second requirement may be satisfied it is enough to notice that if $\delta_n = 0$ then the corresponding $C_1^{(n)}$ has a radius of curvature never less and sometimes greater than r so that C_2 can be placed entirely within $C_1^{(n)}$ by a known theorem.[9] Now let δ_n be fixed satisfying the above requirements and consider the corresponding $C_1^{(n)}$. There are clearly n disjoint θ-intervals in which the radius of curvature (2) of $C_1^{(n)}$ is less than r. Then, by Lemma I, the inner boundary $C_I^{(n)}$ of the vectorial sum $C_1^{(n)}(+)C_2$ will have a corner in the directions (θ_i, θ'_i) for a set of direction intervals including these n disjoint θ-intervals. In general it is not true that two disjoint intervals in which $\rho_1^{(n)}(\theta) \leqq \rho_2(\theta + \pi)$ correspond to distinct corners but in this case the symmetry of $C_1^{(n)}$ makes it obvious that the n intervals mentioned above actually correspond to n distinct corners. Thus, *the inner boundary $C_I^{(n)}$ of the analytic convex sum $C_1^{(n)}(+)C_2$, where $C_1^{(n)}$ is defined by its radius of curvature* (2) *and C_2 is a circle of radius r, has n corners.*

THEOREM II. *There exist convex curves C_1, C_2 whose radii of curvature $\rho_1(\theta)$, $\rho_2(\theta)$ have infinitely many derivatives and which are such that the vector sum $C_1(+)C_2$ has an inner boundary C_I with infinitely many corners.*

Proof. Let C_2 be again a circle of radius r so that $\rho_2(\theta) \equiv r$. Let $\theta_0, \theta_1, \cdots \theta_n, \cdots$ be an infinite sequence of θ-values such that

$$(3) \qquad \pi/2 = \theta_0 > \theta_1 > \cdots > \theta_n \to 0.$$

Let

$$(4) \qquad \rho_1(\theta) \equiv r \text{ if } \theta_{2n} \geqq \theta \geqq \theta_{2n+1}; \qquad (n = 0, 1, 2, \cdots)$$

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[8] W. Blaschke, *Kreis und Kugel*, Leipzig (1916), pp. 115-116.

[9] Cf., e. g., W. Blaschke, *loc. cit.*, p. 116.

and

$$\rho_1(\theta) \equiv r + h_n(\theta) \text{ if } \theta_{2n+1} > \theta > \theta_{2n+2}; \qquad (n = 0, 1, 2, \cdots) \tag{5}$$

where $h_n(\theta) > 0$ is a function [10] (defined only on the interval $\theta_{2n+1} > \theta > \theta_{2n+2}$) for which all derivatives (and the function values) exist, and approach zero as $\theta \to \theta_{2n+1} - 0$ or $\theta \to \theta_{2n+2} + 0$, and such that the first n derivatives are less than $(\theta_{2n+2})^2$ in absolute value. Thus $\rho_1(\theta)$ is defined by (3), (4), (5) for the interval $0 < \theta \leqq \pi/2$. Let the definition of $\rho_1(\theta)$ be completed by the requirement

$$\rho_1(\theta) \text{ is periodic of period } \pi/2. \tag{6}$$

Now $\rho_1(\theta)$ is differentiable infinitely often. This is obvious in the interval $0 < \theta \leqq \pi/2$. Thus, by (6), it is sufficient to show that all derivatives exist at the point $\theta = 0$. But, by (4) for $n = 0$, the left-hand derivatives at $\theta = \pi/2$ and hence, by (6), at $\theta = 0$ are all zero. On the other hand, by (3), (4), (5) and the definition of $h_n(\theta)$, the right-hand derivatives are all zero also. For suppose it has been proved that the k-th derivative at $\theta = 0$ exists and is zero, then the $(k+1)$-th difference quotients are bounded by $(\theta_{2n+2})^2/\theta_{2n+2} \to 0$.

Now, by (6), the closure conditions (1) are obviously satisfied so that there exists a convex curve C_1 of which $\rho_1(\theta)$ is the radius of curvature. The circle C_2 of radius r can be placed entirely within C_1 since $\rho_1(\theta) \geqq r$. Thus the vectorial sum $C_1(+)C_2$ does have an inner boundary curve C_I. But if the "mechanical" interpretation of vectorial addition mentioned above be remembered, it is clear that each of the intervals involved in (4) will be directions of distinct corners of this inner boundary C_I. This completes the proof of Theorem II.

It is noticed that the example was so constructed that the points of C_1 and C_2 which corresponded to a cluster point of corners of C_I were points where the radii of curvature of the two curves were equal. This fact could not be avoided. In fact, it is a direct consequence of Lemma I that if the two curves C_1 and C_2 have continuous radii of curvature $\rho_1(\theta)$ and $\rho_2(\theta)$ and if the inner boundary curve of $C_1(+)C_2$ exists and has infinitely many corners clustering in the direction θ_0 then $\rho_1(\theta_0) = \rho_2(\theta_0 + \pi)$. Thus

THEOREM III. *If C_1 and C_2 have continuous radii of curvature and if the vectorial sum $C_1(+)C_2$ has an inner boundary curve C_I then the corners (if any) of C_I are nowhere dense on the inner curve.*

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[10] Such a function may be taken in the form

$$h_n(\theta) = k_n \exp[(\theta_{2n+1} - \theta)(\theta_{2n+2} - \theta)]$$

where the constant $k_n > 0$ is chosen sufficiently small.

ON THE VALUES OF THE RIEMANN ζ-FUNCTION ON FIXED LINES $\sigma > 1$

By Richard Kershner.

For a fixed $\sigma > 1$, let $M(\sigma)$ denote the closure of the set of values taken by the almost periodic function

$$-\log \zeta(s) = \sum_{n=1}^{\infty} \log\,(1 - p_n^{-\sigma} e^{-it \log p_n}), \qquad -\infty < t < +\infty,$$

where $\zeta(s) = \zeta(\sigma + it)$ is the Riemann ζ-function. It has been shown by Bohr [1] that, due to the linear independence of the logarithms of the prime numbers, $M(\sigma)$ is the vectorial sum[2] of the sequence of curves $S(p_1^{-\sigma})$, $S(p_2^{-\sigma}), \cdots$, where $S(r)$ denotes, for a fixed $r < 1$, the curve

$$(1) \qquad , \quad w = u + iv = \log\,(1 + re^{i\theta}), \qquad 0 \leqq \theta < 2\pi,$$

in the w-plane. Since every $S(r)$ is easily seen to be a convex Jordan curve, it follows from Bohr's general results (cf. [2]) on the vectorial addition of convex curves that $M(\sigma)$ is,[3] for every fixed $\sigma > 1$, either a closed bounded convex region bordered by a convex Jordan curve $A(\sigma)$ or a closed ring-shaped region bordered by two convex Jordan curves $A(\sigma)$ and $B(\sigma)$, where $B(\sigma)$ is entirely within $A(\sigma)$. Following a suggestion of Wintner, Haviland [5] applied the supporting function (Stützfunktion) of Brunn and Minkowski to the study of $A(\sigma)$, using the fact that the supporting function of $A(\sigma)$ is the sum of the supporting functions of all $S(p_n^{-\sigma})$. A corresponding study of the inner curve $B(\sigma)$, in case $B(\sigma)$ exists, has recently been made by the author [8] and results similar to, but more complicated than, those concerning $A(\sigma)$ were obtained. Actually these investigations treated not only the particular problem of $M(\sigma)$ but the general problem of the vectorial addition of any sequence of convex curves. The object of the present paper is to apply

[1] Received October 28, 1936.

[2] By the vectorial sum $S_1(+)\,S_2(+)\cdots(+)\,S_n$ of n sets S_j of complex numbers is meant the set of numbers w representable in at least one way in the form $w = w_1 + w_2 + \cdots + w_n$ where $w_j \subset S_j$. If $n = \infty$, one means by $S_1(+)S_2(+)\cdots$ the set of points w for which $w = w_1 + w_2 + \cdots$ is a convergent series and $w_j \subset S_j$. In the case under consideration ($\sigma > 1$) this series is absolutely convergent for every choice of $w_j \subset S_j$.

[3] For a short proof of this statement cf. Jessen and Wintner [6], p. 69.

these general results to a detailed discussion of $M(\sigma)$ for varying values of σ. Since the closure $N(\sigma)$ of the values attained by $\zeta(\sigma + it)$ on a fixed line $\sigma > 1$ may be obtained from $M(\sigma)$ by a trivial exponential mapping, the results to be obtained may easily be formulated in terms of $\zeta(s)$ instead of $\log \zeta(s)$.

Mention should first be made of the following result of Bohr and Jessen [4]:

(I) *There exists a constant* $\sigma_0 > 1$ *such that* $M(\sigma)$ *is or is not a convex region* (i. e., $B(\sigma)$ *does not or does exist*) *according as* $1 < \sigma \leqq \sigma_0$ *or* $\sigma_0 < \sigma < +\infty$. *This* σ_0 *is the only root* $\sigma > 1$ *of the equation*

$$\text{(2a)} \qquad \arcsin 2^{-\sigma} = \sum_{n=2}^{\infty} \arcsin p_n^{-\sigma}, \text{ so that } \sigma_0 = 1.764\cdots.$$

It is easy to see that $S(r)$ has, for every fixed $r < 1$, the two lines of symmetry $v = 0$ and $u = \frac{1}{2}\log(1 - r^2)$, so that $M(\sigma)$ has, for every fixed $\sigma > 1$, the two lines of symmetry

$$v = 0 \text{ and } u = u_\sigma = \sum_{n=1}^{\infty} \tfrac{1}{2}\log(1 - p_n^{-2\sigma}).$$

It has been shown by Kershner and Wintner [7] that $A(\sigma)$ is, for every $\sigma > 1$, a regular analytic curve and that the same is true of $B(\sigma)$ up to its possible corners, provided that $B(\sigma)$ exists. In view of the fact (cf. (II) below) that $B(\sigma)$ has at most two corners for any σ, this statement implies a remark of Bohr and Jessen [4] to the effect that $B(\sigma)$ contains no straight segments for any σ.

The following theorem will now be deduced:

(II) *There exists a constant* $\sigma^0 > \sigma_0$ *such that* $B(\sigma)$ *has exactly two corners or no corners according as* $\sigma_0 < \sigma < \sigma^0$ *or* $\sigma^0 \leqq \sigma < +\infty$. *This* σ^0 *is the only root* $\sigma > 1$ *of the equation*

$$\text{(2b)} \qquad 2^{-\sigma} = \sum_{n=2}^{\infty} p_n^{-\sigma}, \text{ so that } \sigma^0 = 1.778\cdots.$$

If $\sigma_0 < \sigma < \sigma^0$, *both corners of* $B(\sigma)$ *lie on the real axis.*

This result has been announced, without proof, by Bohr and Jessen at the end of their paper [4], where it is stated that the proof of (II) is similar to that of (I). In attempting to verify (II), it was found that (II) may be deduced from the results of the paper [8] as follows:

Let $P(r)$ denote, for a fixed r in $0 < r < 1$, the circle $|z - 1| = r$, so

that $S(r)$ is, by (1), the image of $P(r)$ under the conformal mapping $u + iv = \log z$. Thus, if $z = \rho e^{i\phi}$, $-\frac{1}{2}\pi \leqq \phi \leqq \frac{1}{2}\pi$, then,

$$u = \log \rho, \qquad v = \phi. \tag{3}$$

The ranges of ρ and ϕ, when z describes $P(r)$, are

$$1 - r \leqq \rho \leqq 1 + r \quad \text{and} \quad -\arcsin r \leqq \phi \leqq \arcsin r.$$

Let $\alpha(r, v)$, for $0 < r < 1$, $0 < v \leqq \arcsin r$, denote the angle measured in the positive direction from the u-axis to the normal to $S(r)$ at that point of $S(r)$ cut by the line of constant ordinate v for which u is greater, and let $\rho(r, \alpha)$ be the radius of curvature of $S(r)$ at this same point. Similarly, let $\alpha_\sigma(v)$ be the angle measured in the positive direction from the u-axis to the normal to $B(\sigma)$ at that point of $B(\sigma)$ cut by the line of constant ordinate v for which u is greater, and let $\rho_\sigma(\alpha)$ be the radius of curvature of $B(\sigma)$ at this point, if the radius of curvature is defined there. Clearly, $0 < \alpha_\sigma(v) \leqq \frac{1}{2}\pi$, for the section of $B(\sigma)$ for which $\alpha_\sigma(v)$ has been defined. Due to the above-mentioned symmetry of $B(\sigma)$ it will be sufficient to consider, in the sequel, only that arc of $B(\sigma)$ for which $0 < \alpha_\sigma(v) \leqq \frac{1}{2}\pi$.

It has been shown in [8] that if $\rho_\sigma(\alpha)$ is defined for a given α in $0 < \alpha \leqq \frac{1}{2}\pi$, then, since $\rho(r, \alpha) = \rho(r, \alpha + \pi)$,

$$\rho_\sigma(\alpha) = \rho(2^{-\sigma}, \alpha) - \sum_{n=2}^{\infty} \rho(p_n^{-\sigma}, \alpha). \tag{4}$$

It was also shown in [8] that, if $\sigma > 1$ is fixed, the radius of curvature $\rho_\sigma(\alpha)$ exists, and so (4) is valid for all positive $\alpha \leqq \frac{1}{2}\pi$ if and only if the expression on the right of (4) is non-negative for all these α and vanishes for isolated values of α, at most. According to Bohr and Jessen, (cf. [3], p. 402),

$$\rho(r, \alpha) = r/\cos v(r, \alpha),$$

where $v(r, \alpha)$ is the unique inverse in $0 < \alpha \leqq \frac{1}{2}\pi$ of $\alpha(r, v)$, so that $0 < v(r, \alpha) \leqq \arcsin r < \frac{1}{2}\pi$. Finally,

$$\cos v(r, \alpha) = (1 - r^2 \sin^2 \alpha)^{\frac{1}{2}}, \tag{5}$$

as seen from (3) and the conformity of the mapping $w = \log z$ (cf. Fig. 1 and Fig. 2 of Bohr and Jessen [4]). Hence, the expression on the right of (4) is

$$f_\sigma(\alpha) = 2^{-\sigma}(1 - 2^{-2\sigma} \sin^2 \alpha)^{-\frac{1}{2}} - \sum_{n=2}^{\infty} p_n^{-\sigma}(1 - p_n^{-2\sigma} \sin^2 \alpha)^{-\frac{1}{2}}, \tag{6}$$

so that, on differentiating with respect to α,

$$f'_\sigma(\alpha) = \sin\alpha\cos\alpha[2^{-3\sigma}(1-2^{-2\sigma}\sin^2\alpha)^{-(3/2)} - \sum_{n=2}^{\infty} p_n^{-3\sigma}(1-2^{-2\sigma}\sin^2\alpha)^{-(3/2)}].$$

Since if $a_n > 0$, then $a_1 \geqq \sum_{n=2}^{\infty} a_n$ implies $a_1^3 \geqq (\sum_{n=2}^{\infty} a_n)^3 > \sum_{n=2}^{\infty} a_n^3$, it follows that $f_\sigma(\alpha) \geqq 0$ implies $f'_\sigma(\alpha) > 0$. Hence, the expression on the right of (4) is a monotone increasing function of α for all those σ for which $f_\sigma(+0) \geqq 0$. Consequently, the radius of curvature $\rho_\sigma(\alpha)$ exists and is represented by (4) for every positive $\alpha \leqq \frac{1}{2}\pi$, if and only if σ is such that the expression on the right of (4) tends to a non-negative limit as $\alpha \to +0$. Now it will be shown that there exists a $\sigma^0 > 1$ such that $f_\sigma(+0)$ is negative, zero or positive according as $1 < \sigma < \sigma^0$, $\sigma = \sigma^0$ or $\sigma^0 < \sigma < +\infty$. First, it is seen from (6) that

$$f_\sigma(+0) = 2^{-\sigma} - \sum_{n=2}^{\infty} p_n^{-\sigma},$$

so that $f_\sigma(+0)$ is positive for large σ, negative for σ near enough to 1 and zero if and only if σ is a root of the equation (2b). Thus, it is sufficient to show that the equation

$$g(\sigma) = 1, \quad \text{where} \quad g(\sigma) = \sum_{n=2}^{\infty} (2/p_n)^\sigma$$

has exactly one root $\sigma = \sigma^0 > 1$. But this is obvious, since $g(\sigma)$ is a continuous, monotone decreasing function for which $g(1+0) = +\infty$, $g(+\infty) = 0$.

In order to complete the proof of (II), it remains to be shown that if $B(\sigma)$ has corners, they lie on the real axis. Suppose, if possible, that $B(\sigma)$ has, for some fixed $\sigma > 1$, a corner not on the real axis. Then it is seen from the double symmetry of $B(\sigma)$ that the α-interval $0 < \alpha \leqq \frac{1}{2}\pi$ contains a subinterval which corresponds to a corner. Let $\alpha_1 \leqq \alpha \leqq \alpha_2$ be this subinterval, so that $0 < \alpha_1 < \alpha_2 \leqq \frac{1}{2}\pi$. It is clear from the considerations of the paper [8], especially from the geometrical construction of $B(\sigma)$ given there, that one can choose this subinterval in such a way that $f_\sigma(\alpha)$ is negative for $\alpha_1 < \alpha < \alpha_2$ and zero for $\alpha = \alpha_1$. Hence $f_\sigma(\alpha)$ is certainly not increasing in the interval $\alpha_1 < \alpha \leqq \frac{1}{2}\pi$. On the other hand, it has been shown above that if $f_\sigma(\alpha_1) = 0$, then $f_\sigma(\alpha)$ is increasing for $\alpha_1 < \alpha \leqq \frac{1}{2}\pi$. This contradiction completes the proof of (II).

Incidentally, the above considerations clearly imply the following result:

(III) *The radius of curvature $\rho_\sigma(\alpha)$ of $B(\sigma)$ is, for every fixed $\sigma > \sigma_0$, defined, continuous and increasing in the α-interval $\alpha_\sigma(+0) < \alpha \leqq \frac{1}{2}\pi$. If $\sigma = \sigma^0$, then $\alpha_\sigma(+0) = 0$ and $\rho_\sigma(+0) = 0$ so that, while the convex curve*

$B(\sigma^0)$ has no corners, its radius of curvature vanishes at the points on the real axis.

If $\sigma \geqq \sigma^0$, then $B(\sigma)$ has no corners by (II), while $B(\sigma)$ does not exist for $\sigma \leqq \sigma_0$ by (I). The following theorem describes the situation in the remaining σ-interval.

(IV) *If $\sigma_0 < \sigma < \sigma^0$ and $\theta = \theta(\sigma)$ denotes the positive angle $(<\pi)$ formed by the two branches of $B(\sigma)$ at a corner of $B(\sigma)$, then $\theta(\sigma)$ is a monotone function of σ and $\theta(\sigma_0 + 0) = 0$, $\theta(\sigma^0 - 0) = \pi$.*

First, it is clear from the definition of $\alpha_\sigma(v)$, as given in the proof of (II), that

$$\theta(\sigma) = \pi - 2\alpha_\sigma(+0). \tag{7}$$

Furthermore, it is clear from the proof of Theorem II_0 in the paper [8] that if Q_a and $P_a{}^n$ denote the points of $B(\sigma)$ and $S(p_n{}^{-\sigma})$ which have the normal inclination α, then Q_a is the vectorial sum of the points $P^1{}_a, P^2{}_{a+\pi}, P^3{}_{a+\pi}, \cdots$. This implies, in view of the definitions of $v_\sigma(\alpha)$ and $v(r, \alpha)$ as given in the proof of (II), that

$$v_\sigma(\alpha) = v(2^{-\sigma}, \alpha) + \sum_{n=2}^{\infty} v(p_n{}^{-\sigma}, \alpha + \pi).$$

Now $v(r, \alpha) = -v(r, \alpha + \pi)$ in view of the summetry of $S(r)$. Consequently, since $\sin v(r, \alpha) = r \sin \alpha$ by (5),

$$v_\sigma(\alpha) = \arcsin (2^{-\sigma} \sin \alpha) - \sum_{n=2}^{\infty} \arcsin (p_n{}^{-\sigma} \sin \alpha).$$

On the other hand it is clear from the definition of α and from the convexity of $B(\sigma)$ that $\alpha_\sigma(v)$ is increasing in the interval $0 < v \leqq \sum_{n=1}^{\infty} \arcsin p_n{}^{-\sigma}$ so that $\alpha_\sigma(+0)$ is the largest root $(< \frac{1}{2}\pi)$ of the equation $v_\sigma(\alpha) = 0$, i. e., of the equation

$$F(\sigma, \alpha) = 0, \tag{8}$$

where

$$F(\sigma, \alpha) = \sum_{n=2}^{\infty} \arcsin (p_n{}^{-\sigma} \sin \alpha) / \arcsin (2^{-\sigma} \sin \alpha) - 1. \tag{9}$$

First, it will be shown by an argument very similar to that of Bohr and Jessen [4], pp. 42-43, that

$$\partial F(\sigma, \alpha)/\partial \alpha < 0, \qquad 0 < \alpha \leqq \tfrac{1}{2}\pi. \tag{10}$$

In fact, it will be shown that each term

$$\phi_n(\sigma, \alpha) = \arcsin (p_n^{-\sigma} \sin \alpha)/\arcsin (2^{-\sigma} \sin \alpha)$$

of (9) has a negative derivative, $\partial\phi_n/\partial\alpha$. Since the logarithmic derivative $\phi_n^{-1}\partial\phi_n/\partial\alpha$ of ϕ_n is the difference between the logarithmic derivatives of the numerator and the denominator, and since $\phi_n > 0$, in order to show that $\partial\phi_n/\partial\alpha < 0$ it is sufficient to prove that the logarithmic derivative $\psi^{-1}\partial\psi/\partial\alpha$ of the function $\psi(\alpha, \lambda) = \arcsin (\lambda \sin \alpha)$ is, for each fixed α in the interval $0 < \alpha \leqq \frac{1}{2}\pi$, an increasing function of λ in $0 < \lambda < 1$. Now the latter logarithmic derivative is

$$(11) \qquad \frac{\lambda \cos \alpha}{\sqrt{1 - \lambda^2 \sin^2 \alpha}\, \arcsin (\lambda \sin \alpha)}, \quad \text{i. e.,} \quad \cot \alpha \frac{\tan y}{y},$$

where $y = \arcsin (\lambda \sin \alpha)$, so that y is an increasing function of λ. Since $(\tan y)/y$ is an increasing function of y in $0 < y \leqq \frac{1}{2}\pi$ it follows that the logarithmic derivative (11) is an increasing function of λ. This completes the proof of (10).

Now it has been pointed out above that $\alpha = \alpha_\sigma(+0)$ is a root of the equation (8). On the other hand, $\alpha = \alpha_\sigma(+0)$ lies in the α-range admitted in (10), since $\alpha_\sigma(+0) = 0$ would imply that $B(\sigma)$ has no corners, whereas $\sigma_0 < \sigma < \sigma^0$ by assumption. On substituting $\alpha = \alpha_\sigma(+0)$ into (8), it follows, from (10) and from the obvious existence of continuous partial derivatives of the function (9), that the function $\alpha_\sigma(+0)$ of σ possesses the continuous derivative

$$(12) \quad d\alpha_\sigma(+0)/d\sigma = -\partial F(\sigma, \alpha)/\partial\sigma : \partial F(\sigma, \alpha)/\partial\alpha \quad (\alpha = \alpha_\sigma(+0)).$$

Now it is easily seen from the considerations of Bohr and Jessen ([4], pp. 42-43) that

$$(13) \qquad \partial F(\sigma, \alpha)/\partial\sigma < 0 \qquad (0 < \alpha \leqq \tfrac{1}{2}\pi,\ \sigma_0 < \sigma < \sigma^0).$$

On comparing (12) with (10) and (13) it follows that $d\alpha_\sigma(+0)/d\sigma < 0$, so that $\theta(\sigma)$ is, in view of (7) an increasing function of σ in the interval $\sigma_0 < \sigma < \sigma^0$. In order to complete the proof of (IV) it is sufficient to notice that $\theta(\sigma_0 + 0) = 0$ is obvious, if one compares the equation (2a) which defines σ_0 with the relations (8), (9), while $\theta(\sigma^0 - 0) = \pi$ similarly follows from the equation (2b) which defines σ^0, if one lets $\alpha \to 0$ in (9).

With regard to the shape of the region $M(\sigma)$ for large values of σ one may say the following:

(V) *The region $M(\sigma)$ is, for large values of σ, asymptotically a circle of radius $2^{-\sigma}$.*

In fact, let $h(r, \alpha)$ be the supporting function of $S(r)$ so that $h(r, \alpha) = u \cos \alpha + v \sin \alpha$, where (u, v) is the point of $S(r)$ for which the normal inclination has the value α. Then on introducing for u and v their expressions in terms of r and α and expanding in powers of r, it may easily be shown that

$$h(r, \alpha) \sim r \text{ as } r \to 0,$$

uniformly in α. Now if $g(\sigma, \alpha)$ is the supporting function of $A(\sigma)$, then, according to the rule of Haviland ([5], p. 333), $g(\sigma, \alpha) = \sum_{n=1}^{\infty} h(p_n^{-\sigma}, \alpha)$, so that

$$g(\sigma, \alpha) \sim 2^{-\sigma} + \sum_{n=2}^{\infty} p_n^{-\sigma} \sim 2^{-\sigma}; \qquad \sigma \to +\infty; \tag{14}$$

uniformly for all α.

Similarly, if $k(\sigma, \alpha)$ is the supporting function of $B(\sigma)$ and if $\sigma > \sigma^0$, then, by [8], p. 743, $k(\sigma, \alpha) = h(2^{-\sigma}, \alpha) - h(p_n^{-\sigma}, \alpha + \pi)$, so that

$$k(\sigma, \alpha) \sim 2^{-\sigma} - \sum_{n=2}^{\infty} p_n^{-\sigma} \sim 2^{-\sigma}, \qquad \sigma \to +\infty; \tag{15}$$

uniformly for all α. Since (14) and (15) hold uniformly for all α, the result (V) follows.

In conclusion, the following obvious remark concerning the shape of $B(\sigma)$, as $\sigma \to \sigma_0$, may be made.

(VI) *The ratio $R(\sigma)$ of the diameter of $B(\sigma)$ along the u-axis to the diameter parallel to the v-axis tends to infinity as $\sigma \to \sigma_0$.*

In fact, a trivial application of the mean value theorem gives $R(\sigma) > \tan \alpha_\sigma(+0)$, while $\tan \alpha_\sigma(+0) \to +\infty$ as $\sigma \to \sigma_0$, by (IV) and (7).

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REFERENCES.

[1] H. Bohr, "Sur la fonction $\zeta(s)$ dans le demi-plan $\sigma > 1$," *Comptes Rendus*, vol. 154 (1912), pp. 1078-1081.

[2] H. Bohr, "Om Addition af uendelig mange konvekse Kurver," *Danske Videnskabernes Selskab, Forhandlinger*, 1913, pp. 325-366.

[3] H. Bohr and B. Jessen, "Om Sandsynlighedsfordelinger ved Addition af konvekse Kurver," *Danske Videnskabernes Selskab*, Skrifter, (8), vol. 12, no. 3 (1929).

[4] H. Bohr and B. Jessen, "On the distribution of the values of the Riemann zeta function," *American Journal of Mathematics*, vol. 58 (1936), pp. 35-45.

[5] E. K. Haviland, "On the addition of convex curves in Bohr's theory of Dirichlet series," *American Journal of Mathematics*, vol. 55 (1933), pp. 332-334.

[6] B. Jessen and A. Wintner, "Distribution functions and the Riemann zeta function," *Transactions of the American Mathematical Society*, vol. 38 (1935), pp. 48-88.

[7] R. Kershner and A. Wintner, "On the boundary of the range of values of $\zeta(s)$," *American Journal of Mathematics*, vol. 58 (1936), pp. 421-426.

[8] R. Kershner, "On the addition of convex curves," *American Journal of Mathematics*, vol. 58 (1936), pp. 737-747.

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